Some variations of Walker's inequality

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Abstract

In this note, we revisit some inequalities involving the elements of a triangle. At first glance, there is nothing in common between them, but actually all of them are equivalent to Walker's inequality. It provides a very convenient and useful quadratic R, r minorant for the square of the semiperimeter s of an acute triangle. Although the problems listed below are borrowed from different sources, all the solutions featured are due to the author of this note.

1 Main results

In what follows, we present some problems involving the circumradius R, the inradius r, and the semiperimeter s of and acute triangle ABC. We begin with an inequality used by Skopets [3].

Problem 1. *Prove that in an acute triangle with side lengths a, b and c, circumradius R and inradius r we have that*

$$a^2 + b^2 + c^2 \geq 4(R + r)^2. \tag{1}$$

Solution. On account of the sine law, $a^2 + b^2 + c^2 \geq 4(R + r)^2$ is equivalent to

$$4R^2(\sin^2 A + \sin^2 B + \sin^2 C) \geq 4R^2\left(1 + \frac{r}{R}\right)^2$$

or

$$\sin^2 A + \sin^2 B + \sin^2 C \geq (\cos A + \cos B + \cos C),^2$$

which is equivalent to

$$\sum_{\text{cyc}} (1 - \cos^2 A) \geq \sum_{\text{cyc}} \cos^2 A + 2 \sum_{\text{cyc}} \cos B \cos C$$

or

$$\sum_{\text{cyc}} (\cos A + \cos B)^2 \leq 3.$$

To prove the above inequality, we use Cauchy's inequality and we have

$$\begin{aligned}(\cos A + \cos B)^2 &\leq (a \cos B + b \cos A)\left(\frac{\cos B}{a} + \frac{\cos A}{b}\right) \\ &= c\left(\frac{\cos B}{a} + \frac{\cos A}{b}\right).\end{aligned}$$

Then,

$$\begin{aligned}&\sum_{\text{cyc}} (\cos A + \cos B)^2 \leq \sum_{\text{cyc}} \left(\frac{c \cos B}{a} + \frac{c \cos A}{b}\right) \\ &= \frac{c \cos B}{a} + \frac{c \cos A}{b} + \frac{a \cos C}{b} + \frac{a \cos B}{c} + \frac{b \cos A}{c} + \frac{b \cos C}{a} \\ &= \sum_{\text{cyc}} \left(\frac{c \cos B + b \cos C}{a}\right) = \sum_{\text{cyc}} \frac{a}{a} = 3. \qquad \square\end{aligned}$$

Remark 1. *During the above solution, we obtained the inequality*

$$\sum_{\text{cyc}} (\cos A + \cos B)^2 \leq 3 \tag{2}$$

which is a trigonometric equivalent of the inequality claimed in Problem 1 [1]. It holds in any acute triangle ABC*.*

Remark 2. *Since* $a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2)$*, we have that* $a^2 + b^2 + c^2 \geq 4(R + r)^2 \iff 2(s^2 - 4Rr - r^2) \geq 4(R + r)^2$ *or, equivalently,*

$$s^2 \geq 2R^2 + 8Rr + 3r^2. \tag{W}$$

*This is known as **Walker's Inequality** for an acute triangle.*

Problem 2. *Prove that the following inequality holds in any triangle:*

$$\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^2+\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2+\left(\sin\frac{C}{2}+\sin\frac{A}{2}\right)^2\le 3. \tag{3}$$

Solution. Since dealing with the sine of a half angle is not very convenient, we will use the cosine theorem to obtain a metrical representation for $\sin\frac{A}{2}, \sin\frac{B}{2}, \sin\frac{C}{2}$ and also a metrical equivalent of the inequality claimed in the statement. Namely,

$$c^2=a^2+b^2-2ab\cos C=(a-b)^2+4ab\sin^2\frac{C}{2},$$

from which it follows that

$$\sin^2\frac{C}{2}=\frac{(s-a)(s-b)}{ab}.$$

Likewise, we get

$$\sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc}$$

and

$$\sin^2\frac{B}{2}=\frac{(s-a)(s-c)}{ac}.$$

Then, by Cauchy's inequality, we have

$$\begin{aligned}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^2&=\frac{s-c}{c}\left(\sqrt{s-b}\cdot\frac{1}{\sqrt{b}}+\sqrt{s-a}\cdot\frac{1}{\sqrt{a}}\right)^2\\&\le\frac{s-c}{c}\cdot\left(\frac{1}{a}+\frac{1}{b}\right)(s-a+s-b)\\&=\frac{(a+b-c)(a+b)c}{2abc}\\&=\frac{(a^2+b^2)c-c^2(a+b)+2abc}{2abc}\\&=1+\frac{(a^2+b^2)c-c^2(a+b)}{2abc}\end{aligned}$$

and, therefore,

$$\begin{aligned}\sum_{\text{cyc}}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^2 &\le 3+\sum_{\text{cyc}}\frac{(a^2+b^2)c-c^2(a+b)}{2abc}\\ &= 3+\frac{1}{2abc}\left(\sum_{\text{cyc}}(a^2+b^2)c-\sum_{\text{cyc}}c^2(a+b)\right)\\ &= 3.\end{aligned}$$

□

Remark 3. *We have*

$$\sum_{\text{cyc}}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^2\le 3 \iff \left(\sum_{\text{cyc}}\sin\frac{A}{2}\right)^2\le\sum_{\text{cyc}}\left(1-\sin^2\frac{A}{2}\right),$$

which is equivalent to the inequality

$$\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2\le\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2} \qquad (4)$$

presented by Andreescu and Dospinescu [2].

Remark 4. *We have that*

$$\sum_{\text{cyc}}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^2\le 3 \iff 2\sum_{\text{cyc}}\sin^2\frac{A}{2}+2\sum_{\text{cyc}}\sin\frac{A}{2}\sin\frac{B}{2}\le 3$$

and

$$\begin{aligned}&\sum_{\text{cyc}}\sin\frac{A}{2}\sin\frac{B}{2}\le\sum_{\text{cyc}}\left(1-2\sin^2\frac{A}{2}\right)\\ \iff\ & 2\sum_{\text{cyc}}\sin\frac{A}{2}\sin\frac{B}{2}\le\sum_{\text{cyc}}\cos A,\end{aligned}$$

from which it follows that

$$\sum_{\text{cyc}}\sin\frac{A}{2}\sin\frac{B}{2}\le\frac{1}{2}\left(1+\frac{r}{R}\right). \qquad (5)$$

Remark 5. *Let* $\alpha = \dfrac{\pi - A}{2}$, $\beta = \dfrac{\pi - B}{2}$ *and* $\gamma = \dfrac{\pi - C}{2}$. *Since* $\alpha, \beta, \gamma \in (0, \pi/2) > 0$ *and* $\alpha+\beta+\gamma = \pi$ *and, therefore, there exists an acute triangle with angles* α, β, γ, *which has side lengths* a, b, c, *circumradius* R *and inradius* r *for some values of* a, b, c, R, r. *Then, inequality* (4) *becomes*

$$\begin{aligned} &\left(\sum_{\text{cyc}} \cos\left(\frac{\pi}{2} - \frac{A}{2}\right)\right)^2 \le 2\sum_{\text{cyc}} \sin^2\left(\frac{\pi}{2} - \frac{A}{2}\right) \\ \iff &\left(\sum_{\text{cyc}} \cos\alpha\right)^2 \le 2\sum_{\text{cyc}} \sin^2\alpha \\ \iff &\left(1 + \frac{r}{R}\right)^2 \le 2\sum_{\text{cyc}} \frac{a^2}{4R^2} \iff (1). \end{aligned}$$

Problem 3. *Let* $x, y, z > 0$ *be any real numbers such that* $x^2 + y^2 + z^2 + 2xyz = 1$. *Prove that*

$$x^2 + y^2 + z^2 + xy + yz + zx \le 3/2. \tag{6}$$

Solution. We have that

$$x^2 + y^2 + z^2 + xy + yz + zx \le 3/2 \iff \sum_{\text{cyc}} (x + y)^2 \le 3.$$

Since any positive solution (x, y, z) of the equation $x^2 + y^2 + z^2 + 2xyz = 1$ can be represented as

$$(x, y, z) = \left(\sqrt{\frac{bc}{(a+b)(c+a)}}, \sqrt{\frac{ca}{(b+c)(a+b)}}, \sqrt{\frac{ab}{(c+a)(b+c)}}\right)$$

for all $a, b, c > 0$, then, using Cauchy's inequality, we obtain

$$\begin{aligned}\sum_{\text{cyc}}(x+y)^2 &= \sum_{\text{cyc}}\left(\sqrt{\frac{bc}{(a+b)(c+a)}}+\sqrt{\frac{ca}{(b+c)(a+b)}}\right)^2\\ &= \sum_{\text{cyc}}\frac{c}{a+b}\left(\sqrt{\frac{b}{c+a}}+\sqrt{\frac{a}{b+c}}\right)^2\\ &= \sum_{\text{cyc}}\frac{c}{a+b}\left(\sqrt{b}\cdot\frac{1}{\sqrt{c+a}}+\sqrt{a}\cdot\frac{1}{\sqrt{b+c}}\right)^2\\ &\le \sum_{\text{cyc}}\frac{c}{a+b}(b+a)\cdot\left(\frac{1}{c+a}+\frac{1}{b+c}\right)\\ &= \sum_{\text{cyc}}\left(\frac{c}{c+a}+\frac{c}{b+c}\right) = 3. \qquad \square\end{aligned}$$

Remark 6. *Taking into account that*

$$\begin{aligned}&\{(x,y,z) \mid x,y,z>0 \text{ and } x^2+y^2+z^2+2xyz=1\}\\ =&\{(\cos\alpha,\cos\beta,\cos\gamma)\mid x,y,z>0, \alpha,\beta,\gamma\in(0,\pi/2), \alpha+\beta+\gamma=\pi\},\end{aligned}$$

we can rewrite inequality (6) *as*

$$\begin{aligned}&\sum_{\text{cyc}}\cos^2\alpha+\sum_{\text{cyc}}\cos\alpha\cos\beta\le 3/2\\ \iff\ & 2\sum_{\text{cyc}}\cos^2\alpha+2\sum_{\text{cyc}}\cos\alpha\cos\beta\le 3 \iff (2).\end{aligned}$$

(Here, numbers α, β and γ are interpreted as angles of an acute triangle.)

Finally, up to notations and interpretations, here is the chain of equivalent inequalities:

$$\text{(W)} \iff (1) \iff (2) \iff (3) \iff (4) \iff (5) \iff (6).$$

We conclude that a proof of any of them is at the same time a proof for all others.

References

[1] Andreescu, T. and Dospinescu, G. “Problem 12”. *Problems from the Book*. XYZ Press, 2008, p. 22.

[2] Andreescu, T. and Dospinescu, G. “Problem 9”. *Problems from the Book*. XYZ Press, 2008, p. 21.

[3] Skopets, Z. A. “Problem M1178”. *Kvant* (1989), p. 153.

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